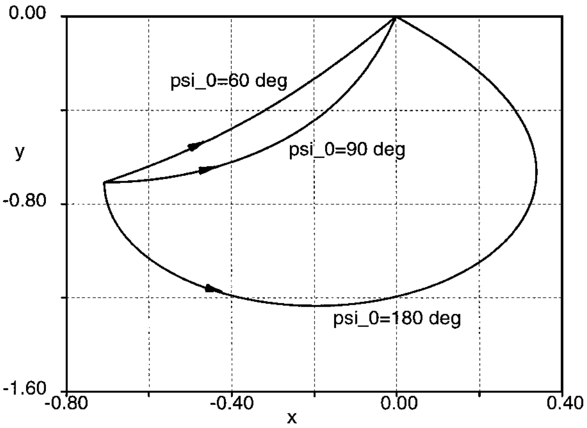

Fig. 2 Trajectories with different initial $\psi(0)$.

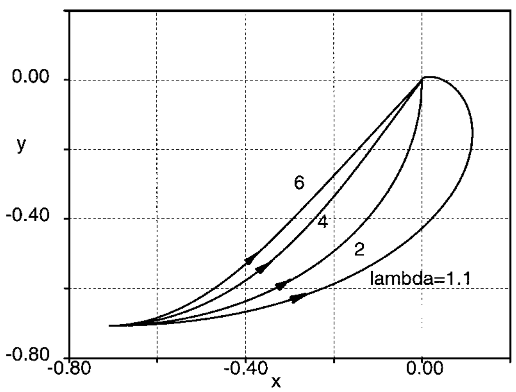

Fig. 3 Trajectories with different λ values.

Remarks:

1) Although theoretically r will not reach zero in finite time (except for $\lambda = 2$), simulations reveal that r always reaches a very small value (practically zero) in finite times.

2) The property that $\dot{\psi} = \lambda\dot{\psi}^* \to 0$ for $\lambda > 2$ is desirable. This means that the required turning acceleration of the interceptor in the horizontal plane, $V_T\dot{\psi}$, is approaching zero near the end.

3) Similar results are obtained in Ref. 7 for constant-velocity interceptor and target. Specifically, it can be deduced from Ref. 7 that $\lambda > 1$ is required for a constant-velocity interceptor to capture a target if the target is stationary. Also, applying the discussion in Ref. 7 to a nonmoving target yields that for $\lambda > 2$ the line-of-sight (LOS) rotation rate is decreasing near the end, although the conclusion of the vanishing LOS rotation rate is not readily available from Ref. 7.

4) The conclusions obtained in this Note are independent of the actual variations of the interceptor velocity $V_T(t)$ and flight-path angle $\gamma(t)$. This enables the guidance algorithm in the vertical direction, which will result in changes in γ, to function independently.

Figure 2 shows the intercept trajectories based on Eqs. (8–10) for $\lambda = 2.5$ with different $\psi(0)$ values and fixed dimensionless $r(0) = 1$ and $\theta(0) = 225$ deg, where x and y are normalized by $r_0 = \sqrt{[x^2(0) + y^2(0)]}$. Figure 3 contains the comparison of intercept trajectories for the fixed initial conditions $r(0) = 1$, $\theta(0) = 225$ deg, and $\psi(0) = 90$ deg as λ varies. It is seen that, the larger λ is, the more rapidly the trajectory will approach a direct collision course.

III. Conclusions

A PN guidance law is applied to an interceptor that has arbitrary time-varying velocity and is to reach a nonmoving target point. When the proportional constant is greater than unity, intercept will occur for all initial conditions, except for the case when the interceptor is moving away from the target initially along the LOS. When the proportional constant is greater than 2, the intercept will end in a direct collision course, with the required normal acceleration being zero for the interceptor.

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Matrix Symmetrization

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Introduction

THE role of Riccati and Lyapunov equations and their solutions in optimal control and Kalman filtering is well known.[1–3] It is also well known that the solution of these equations, which is a real symmetric matrix, loses its symmetry because of numerical errors. The simplest and most straightforward way to restore this property is symmetrization, which is performed at each instant in which the matrix is computed[4–6]; that is, if we denote by P the raw solution at a certain stage, then P is replaced by the matrix P_s, where

$$P_s = (P + P^T)/2 \tag{1}$$

In the ensuing we will refer to P_s as the symmetrized P.

Whereas the replacement of P by P_s is the obvious way to solve the problem, there may seemingly be better ways to restore symmetry. In particular, one may wish to replace P by the symmetric matrix that is the closest to it. This raises the question, what is the symmetric matrix that is the closest, in the Frobenius norm, to P? It turns out that the real symmetric matrix closest to P is the symmetrized P. Although proven in 1955 (Ref. 7), this fact is not widely known. The purposes of this Note are, first, to bring this fact to the attention of the readers; second, to present a new proof that is different from that presented in Ref. 7; and finally, to show that this result can be explained in a rather simple manner.

Closest Symmetric Matrix

Next we prove the fact that P_s is P_c, where P_c denotes the $n \times n$ symmetric matrix closest to the $n \times n$ P matrix. The following theorem and proof are according to Ref. 7.

Theorem: The closest symmetric matrix to a real matrix, in Frobenius (Euclidean) norm, is its symmetrized matrix.

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Proof: Let S be any $n \times n$ real symmetric matrix. Write the identity

$$P - \frac{P + P^T}{2} = \frac{P - S}{2} - \frac{P^T - S}{2} = \frac{P - S}{2} - \frac{(P - S)^T}{2} \quad (2)$$

When applying the triangle inequality of the norm to this identity, and noting that $\|(P - S)\| = \|(P - S)^T\|$ (all through this Note, $\|\cdot\|$ symbolizes the Frobenius norm), one obtains

$$\left\| P - \frac{P + P^T}{2} \right\| \leq \left\| \frac{P - S}{2} \right\| + \left\| \frac{(P - S)^T}{2} \right\| = \|P - S\| \quad (3)$$

Then using Eq. (1) the latter result can be written as

$$\|P - P_s\| \leq \|P - S\| \quad (4)$$

Because S is any symmetric matrix, it follows that there is no symmetric matrix that is closer to P than P_s, and hence $P_s = P_c$. □

The preceding proof is an algebraic one. Next we introduce a new calculus-based proof. As before, let S be any real symmetric matrix. Define $\boldsymbol{J}$ as $\|P - S\|^2$; that is,

$$\boldsymbol{J} = \mathrm{tr}\{(P - S)(P - S)^T\} = \mathrm{tr}\{(P - S)(P^T - S)\} \quad (5)$$

We wish to find that S that is the closest to P. Let us express S as

$$S = P_c + \varepsilon H \quad (6)$$

where ε is a scalar. When this expression for S is substituted into Eq. (5), we obtain

$$\boldsymbol{J} = \mathrm{tr}\big\{(P - P_c - \varepsilon H)(P^T - P_c - \varepsilon H)\big\} \quad (7)$$

A necessary condition for $\boldsymbol{J}$ to be minimum is

$$\left.\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}\varepsilon}\right|_{\varepsilon = 0} = 0 \quad (8)$$

where H is an $n \times n$ symmetric matrix [imposed by Eq. (6)] chosen such that this derivative exists but otherwise is arbitrary. Application of this condition to $\boldsymbol{J}$ of Eq. (7) yields

$$\left.\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}e}\right|_{\varepsilon = 0} = \mathrm{tr}\big\{-H\big(P^T - P_c\big) - \big(P - P_c\big)H\big\} = 0 \quad (9)$$

Using the properties $\mathrm{tr}\{A + B\} = \mathrm{tr}\{A\} + \mathrm{tr}\{B\}$ and $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$, we obtain, from Eq. (9),

$$\mathrm{tr}\big\{\big(P^T - P_c + P - P_c\big)H\big\} = 0 \quad (10)$$

Because this holds for any symmetric matrix H for which Eq. (8) holds, the expression in parentheses must be zero. This yields

$$P_c = (P + P^T)/2 = P_s \quad (11)$$

It is easy to see that this is a point of minimum. □

The result $P_c = (P + P^T)/2$ is not at all surprising. To see this, let $p_{c,i,j}$ be the i,j element of P_c, and similarly, let $p_{i,j}$ be the i,j element of P. Obviously, the closest to the i,i element of P in the symmetric matrix P_c is $p_{i,i}$ itself; that is, $p_{c,i,i} = p_{i,i}$, which is identical to writing

$$p_{c,i,i} = \frac{p_{i,i} + p_{i,i}}{2} \quad (12)$$

Next we want to determine $p_{c,i,j}$ when $i \neq j$. Note that, for $S = P_c$, $\boldsymbol{J}$ defined in Eq. (5) is equal to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (p_{i,j} - p_{c,i,j})^2$$

For each i and j, this sum includes the quantity $\boldsymbol{d}^2$ defined as $\boldsymbol{d}^2 = (p_{i,j} - p_{c,j,i})^2 + (p_{i,j} - p_{c,j,i})^2$, which is contributed by the elements of the difference matrix $\Delta = (P - P_c)$ that are symmetric about the diagonal of Δ. Because P_c is symmetric, we can write $p_{c,i,j} = p_{c,j,i} = x$. Because P_c is the symmetric matrix closest to P, x minimizes $\boldsymbol{d}^2$. It is easy to show that $\boldsymbol{d}^2$ is minimal when

$$x = \frac{p_{i,j} + p_{j,i}}{2} \quad (13)$$

Equations (12) and (13) express the fact that P_c is identical to P_s.

Conclusions

In this Note we pointed out that the symmetrized real matrix is also the symmetric matrix that is the closest, in the Euclidean norm, to the matrix being symmetrized. This implies that, when symmetrizing the solutions to Riccati and Lyapunov equations, one actually replaces the solution by its closest symmetric matrix. A proof of this fact that followed a proof given in the literature in 1955 was presented. A new, calculus-based, proof was also introduced. It was shown that this result can be obtained using simple rationale.

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Translational Motion Control of Vertical Takeoff Aircraft Using Nonlinear Dynamic Inversion

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Nomenclature

$a_{x\,\mathrm{dem}}, a_{y\,\mathrm{dem}}, a_{z\,\mathrm{dem}}$	= translational acceleration commands in Earth/wind x, y, and z directions, respectively
F_x, F_y, F_z	= body axis forces in x, y, and z directions, respectively
g	= gravity
m	= aircraft mass
p, q, r	= body axis roll, pitch, and yaw rate, respectively
u, v, w	= axial, lateral, and normal components of aircraft velocity relative to the air, respectively
V_{bw}	= acceleration to velocity transformation bandwidth

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